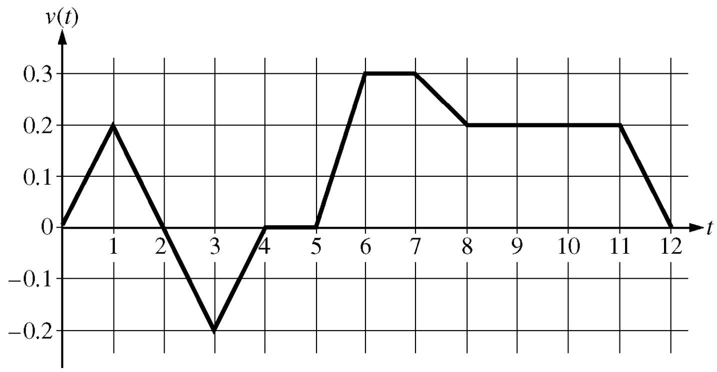
AP® CALCULUS AB
2009 SCORING GUIDELINES

Question 1



Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

(a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_0^{12} |v(t)|\, dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|\, dt$.

(c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

(d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function w given by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

(a) $a(7.5) = v'(7.5) = \dfrac{v(8) - v(7)}{8 - 7} = -0.1$ miles/minute^2	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{units} \end{cases}$
(b) $\int_0^{12} |v(t)|\, dt$ is the total distance, in miles, that Caren rode during the 12 minutes from $t = 0$ to $t = 12$. $\int_0^{12} |v(t)|\, dt = \int_0^{2} v(t)\, dt - \int_2^{4} v(t)\, dt + \int_4^{12} v(t)\, dt$ $= 0.2 + 0.2 + 1.4 = 1.8$ miles	$2 : \begin{cases} 1 : \text{meaning of integral} \\ 1 : \text{value of integral} \end{cases}$
(c) Caren turns around to go back home at time $t = 2$ minutes. This is the time at which her velocity changes from positive to negative.	$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{reason} \end{cases}$
(d) $\int_0^{12} w(t)\, dt = 1.6$; Larry lives 1.6 miles from school. $\int_0^{12} v(t)\, dt = 1.4$; Caren lives 1.4 miles from school. Therefore, Caren lives closer to school.	$3 : \begin{cases} 2 : \text{Larry's distance from school} \\ \quad 1 : \text{integral} \\ \quad 1 : \text{value} \\ 1 : \text{Caren's distance from school} \\ \quad \text{and conclusion} \end{cases}$

1A1

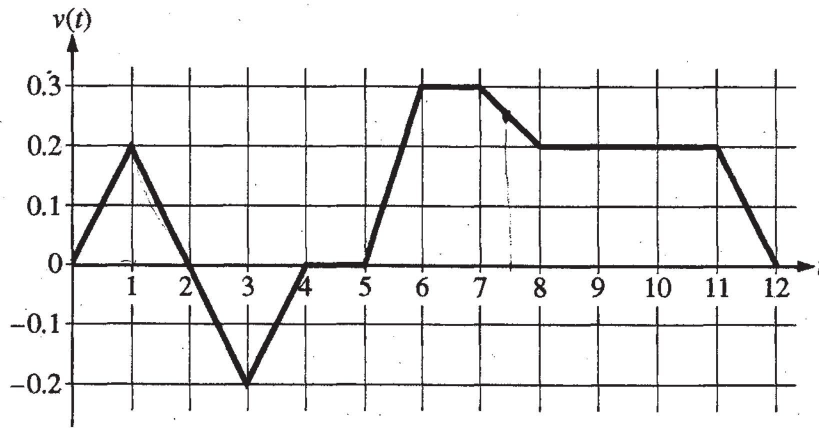
CALCULUS BC

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



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Work for problem 1(a)

$a(7.5) = v'(7.5) = \dfrac{v(8)-v(7)}{8-7} = 0.2 - 0.3 = -0.1$

$\therefore a(7.5) = -0.1 \text{ mi}/\text{min}^2$

Work for problem 1(b)

$\int_0^{12} |v(t)|\, dt = 0.2 + 0.2 + 0.15 + 0.3 + 0.2 + 0.05 + 6(0.1) + 0.1$

$= 1.8$ miles

$\therefore \int_0^{12} |v(t)|\, dt$ is the total distance that Caren traveled from time $t=0$ min to $t=12$ min to arrive to the school, which is 1.8 mi.

Continue problem 1 on page 5

1A₂

Work for problem 1(c)

She turns around at $t = 2$ minutes because that is when her velocity changes from positive to negative.

Work for problem 1(d)

$\int_0^{12} w(t)\,dt = \int_0^{12} \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)dt = 1.6$ mi

The distance from Larry's house to school : 1.6mi

$\int_0^{12} v(t)\,dt = 0.15+0.3+0.2+0.05+0.6+0.1 = 1.4$ mi

The distance from Caren's house to school : 1.4mi

∴ Caren lives closer to school because the distance from school to her house is smaller than that to Larry's house.

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1B1

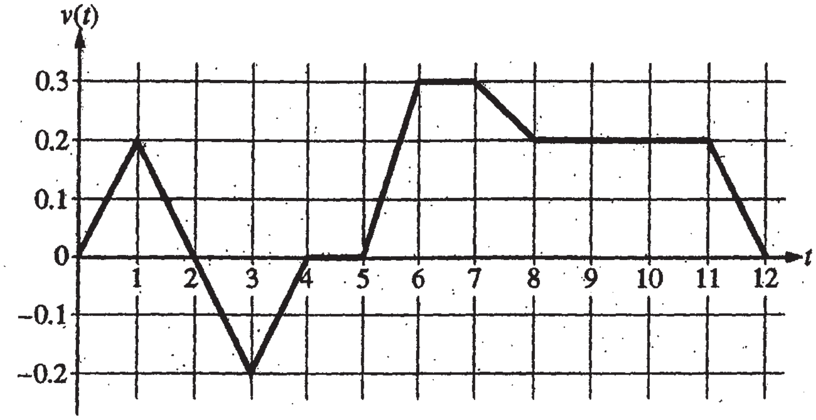
CALCULUS AB

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



Work for problem 1(a)

$a(t)\ \frac{dv}{dt}\ \frac{v_f - v_i}{t_f - t_i}$ $\frac{.2-.3}{8-7}$

$\frac{-.1}{1}$ $-.1$

$-.1$ miles per minute

Work for problem 1(b)

$\int_0^{12} |v(t)|\,dt$ would show the total distance in miles that Caren traveled in 12 minutes

$\int_0^2 v(t)dt - \int_2^4 v(t)dt + \int_4^{12} v(t)dt$ $\frac{.3}{2} + .11 + .1 + .05$

~~.2~~ $.2 + .41$ $.81$

Continue problem 1 on pag

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Work for problem 1(c)

$t=2$ because her velocity changes from + to − and the $\int_0^2 |v(t)|\,dt = \int_2^4 |v(t)|\,dt$

Work for problem 1(d)

$w(t) = \frac{\pi}{15} \sin\left(\frac{\pi}{12} t\right)$

Caren $\int^{12} |v(t)|\,dt = .81$ miles

Larry $\int_0^{12} w(t)\,dt = 1.6$ miles

Caren lives closer to school because she traveled less distance to get there

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1C1

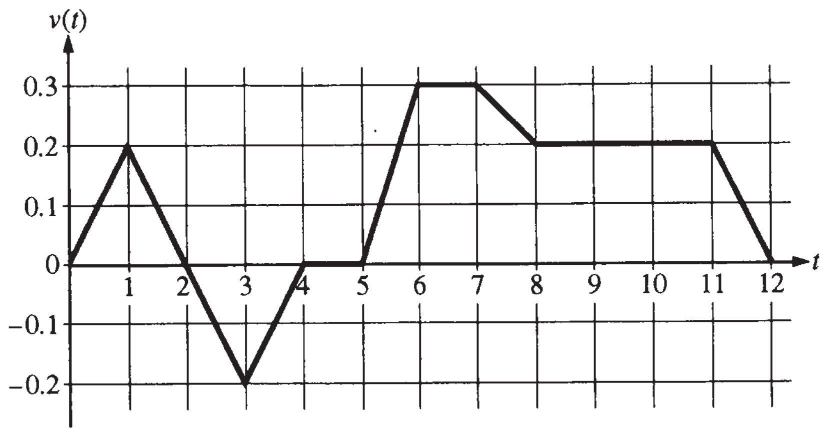
CALCULUS BC

SECTION II, Part A

Time—45 minutes

Number of problems—3

A graphing calculator is required for some problems or parts of problems.



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Work for problem 1(a)

acceleration = $v'(t)$.

$v'(7.5)$

Work for problem 1(b)

$\int$ of velocity is the position.

so, $\int_0^{12}|v(t)|dt$ means ~~the~~ how much or how far of distance Caren rode ~~bi~~ bicycle in 12 minutes.

$$\int_0^{12}|v(t)\,dt = \cancel{\frac{(1\times0.2)}{2} + \frac{(1\times0.2)}{2} +}$$

↑ signed area.

$$\left[\left[\frac{(1\times0.2)}{2}\times4\right] + \left(\frac{1\times0.3}{2}\right) + (1\times0.3) + (1\times0.2) + \left(\frac{1\times0.1}{2}\right) + ((1\times0.2)\times3) + \left(\frac{1\times0.2}{2}\right)\right] = 1.8$$

Continue problem 1 on page 5

Work for problem 1(c)

She ~~she~~ turns around to go back home at $t=2$.

the graph in the interval $1 \le x \le 3$, changes the direction.

~~she was~~

Work for problem 1(d)

~~At t=6, w(t) = $\frac{\pi}{15}\sin\left(\frac{\pi}{2}\right) = \frac{\pi}{15} = 0.20930$.~~

~~Larry's position from home~~

Larry's ~~po~~ distances to school : $\int_0^{12} \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)dt = 1.6$.

Caren's " : $\int_0^{12} f(t)\,dt = 1.8$. (from answer (b))

so. Larry lives closer to school.

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Question 1

Overview

This problem opened with a piecewise-linear graph. The graph models the velocity function $v(t)$ for bicycle rider Caren during a 12-minute period in which she travels along a straight road, starting at home at time $t = 0$ and arriving at school at time $t = 12.$ Part (a) asked for Caren's acceleration at a particular time during her trip, which required students to recognize that acceleration is the derivative of velocity and to acquire the value of this derivative from the slope of the appropriate line segment on the given velocity graph. Part (b) asked for an interpretation of $\int_0^{12} |v(t)|\, dt$ in terms of Caren's trip, as well as for the value of this integral. Part (c) provided the additional information that Caren needed to return home to retrieve her homework shortly after starting her journey. Students needed to associate Caren's direction of motion with the sign of her velocity to determine at what time she turned around. (Students were not required to observe that the distances traveled in each direction match.) In part (d) the velocity function for another bicycle rider, Larry, was modeled by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$ for the same 12-minute period, $0 \le t \le 12.$ This part asked who lives closer to school, Caren or Larry. To respond, students needed to compute the two home-to-school distances, $\int_0^{12} v(t)\, dt$ (which equals $\int_5^{12} v(t)\, dt$) and $\int_0^{12} w(t)\, dt.$

Sample: 1A
Score: 9

The student earned all 9 points.

Sample: 1B
Score: 6

The student earned 6 points: 1 point in part (a), 1 point in part (b), 2 points in part (c), and 2 points in part (d). In part (a) the student earned the first point for evaluating a correct difference quotient. The student's units of miles per minute are incorrect. In part (b) the student earned the first point for a correct interpretation of the meaning of the integral using correct units. The student's evaluation of the integral is incorrect. In part (c) the student's work is correct. The statement regarding the integrals of $|v(t)|$ on the two different intervals is correct but was not required to earn the point. In part (d) the student earned 2 points for Larry's distance from school by stating and evaluating the correct definite integral. The student's value for Caren's distance from school is incorrect, so the last point was not earned.

Sample: 1C
Score: 4

The student earned 4 points: no points in part (a), 1 point in part (b), 1 point in part (c), and 2 points in part (d). In part (b) the student does not indicate the correct units of miles, so the first point was not earned. The student earned the second point for a correct evaluation of the integral. In part (c) the student earned the first point for a correct answer. The student's reason is not valid. In part (d) the student earned 2 points for Larry's distance from school by stating and evaluating the correct definite integral. The student's value for Caren's distance from school is incorrect, so the last point was not earned.

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2009 SCORING GUIDELINES

Question 2

The rate at which people enter an auditorium for a rock concert is modeled by the function R given by $R(t) = 1380t^2 - 675t^3$ for $0 \le t \le 2$ hours; $R(t)$ is measured in people per hour. No one is in the auditorium at time $t = 0,$ when the doors open. The doors close and the concert begins at time $t = 2.$

(a) How many people are in the auditorium when the concert begins?

(b) Find the time when the rate at which people enter the auditorium is a maximum. Justify your answer.

(c) The total wait time for all the people in the auditorium is found by adding the time each person waits, starting at the time the person enters the auditorium and ending when the concert begins. The function w models the total wait time for all the people who enter the auditorium before time t. The derivative of w is given by $w'(t) = (2 - t)R(t).$ Find $w(2) - w(1),$ the total wait time for those who enter the auditorium after time $t = 1.$

(d) On average, how long does a person wait in the auditorium for the concert to begin? Consider all people who enter the auditorium after the doors open, and use the model for total wait time from part (c).

(a) $\int_0^2 R(t)\,dt = 980$ people

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(b) $R'(t) = 0$ when $t = 0$ and $t = 1.36296$

The maximum rate may occur at 0, $a = 1.36296,$ or 2.

$R(0) = 0$
$R(a) = 854.527$
$R(2) = 120$

The maximum rate occurs when $t = 1.362$ or 1.363.

$3 : \begin{cases} 1 : \text{considers } R'(t) = 0 \\ 1 : \text{interior critical point} \\ 1 : \text{answer and justification} \end{cases}$

(c) $w(2) - w(1) = \int_1^2 w'(t)\,dt = \int_1^2 (2 - t)R(t)\,dt = 387.5$

The total wait time for those who enter the auditorium after time $t = 1$ is 387.5 hours.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

(d) $\frac{1}{980} w(2) = \frac{1}{980}\int_0^2 (2 - t)R(t)\,dt = 0.77551$

On average, a person waits 0.775 or 0.776 hour.

$2 : \begin{cases} 1 : \text{integral} \\ 1 : \text{answer} \end{cases}$

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Work for problem 2(a)

$\int_0^2 R(t)\,dt = \boxed{980 \text{ people}}$

Work for problem 2(b)

$\underline{R'(t)=0}$

$t=1.3629=A$

$\underline{\text{Endpoints}}$

$t=0$

$t=2$

R has an absolute maximum at $t=1.3629$ hours on $t\in[0,2]$ guaranteed by the EVT.

t	$R(t)$
0	0
A	854.5273
2	120

Continue problem 2 on page 7.

Work for problem 2(c)

$\int_1^2 w'(t)\,dt = \boxed{387.5 \text{ hours}}$

Work for problem 2(d)

$\int_0^2 w'(t)\,dt \div \int_0^2 R(t)\,dt = \frac{760}{980} = \boxed{0.7755 \frac{\text{hours}}{\text{person}}}$

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Work for problem 2(a)

People $= \int_0^2 R(t)\,dt = 980$ people

There are approximately 980 people in the auditorium when the concert begins.

Work for problem 2(b)

$R(t) = 1380t^2 - 675t^3$

$R'(t) = 2760t - 2025t^2$

$0 = 2760t - 2025t^2$

$t = 0, 1.363$

$R'(t)$ ↘ 0 ↗ 0 ↘

-1, 0, 1, 1.363, 2

$R'(-1) = -4785$ $R'(2) = -2580$

$R'(1) = 735$

The rate at which people enter the auditorium is at a max at $t = 1.363$ hour b/c $R'(t) = 0$ at $t = 1.363$ hours & changes from + to -

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Continue problem 2 on page 7.

Work for problem 2(c)

$\int_a^b w'(t) = w(b) - w(a)$

$w'(t) = (2-t)R(t)$

$w(2) - w(1) = \int_1^2 (w'(t))\,dt$

$= \int_1^2 (2-t)(1380t^2 - 675t^3)\,dt$

$= 387.5$ hours

Total wait time is 387.5 hours according to the fundamental theorem of Calculus

Work for problem 2(d)

$\text{Avg. wait time} = \frac{1}{b-a}\int_a^b w'(t)\,dt$

$= \frac{1}{2-1}\int_1^2 w'(t)\,dt$

$= \frac{1}{1}\int_1^2 w'(t)\,dt$

$= 387.5$

On average a person waits 387.5 hours in the auditorium for concert to begin

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Work for problem 2(a)

$$\int_0^2 1380t^2 - 675t^3\,dt$$

$$1380 \cdot \frac{1}{3} t^3 - 675 \cdot \frac{1}{4} t^4$$

$$460t^3 - 168.75t^4$$

$$\left(460(2)^3 - 168.75(2)^4\right) - \left(460(0)^3 - 168.75(0)^4\right)$$

$980 - 0 = 980$ people are in the auditorium when the concert begins

Work for problem 2(b)

$$R(t) = 1380t^2 - 675t^3$$

$$R'(t) = 2760t - 2025t^2$$

$$2760t - 2025t^2 = 0$$

$$t(2760 - 2025t) = 0$$

$$t = 0, \ 135/184$$

$$2760 - 2025t = 0$$

$$-2025t = -2760$$

$$t = 135/184$$

–	min	+	max	–
	0		135/184	

The rate at which people enter the auditorium is at its maximum at time 135/184.

Continue problem 2 on page 7.

Work for problem 2(c)

$w'(t) = (2-t)\,R(t)$

$w'(t) = (2-t)(1380t^2 - 675t^3)$

$\int_1^2 (2-t)(1380t^2 - 675t^3)\,dt =$ ______ hours

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Work for problem 2(d)

$$\frac{\text{\_\_\_\_\_\_ hours = the total wait time}}{\text{980 people who are in the auditorium when the concert begins}}$$

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AP® CALCULUS AB
2009 SCORING COMMENTARY

Question 2

Overview

This problem presented students with a polynomial $R(t) = 1380t^2 - 675t^3$ that modeled the rate, in people per hour, at which people enter an auditorium during the two hours ($0 \le t \le 2$) prior to the start of a rock concert. It was stated that the auditorium was empty at time $t = 0$, and part (a) asked for the number of people in the auditorium at time $t = 2$, which required computation of the definite integral $\int_0^2 R(t)\,dt$. In part (b) students needed to find the time t that maximizes $R(t)$. Part (c) defined the total wait time for all the people in the auditorium and stated that a function w that models the total wait time for all the people who entered the auditorium by time t has derivative $w'(t) = (2 - t)R(t)$. Students were asked to evaluate $w(2) - w(1)$ and should have recognized that this is computed by $\int_1^2 w'(t)\,dt$. Part (d) asked for the average amount of time that a concertgoer spent waiting for the concert to begin after entering the auditorium. Students needed to compute the total wait time, $\int_0^2 w'(t)\,dt$, for all people attending the concert and divide this by the number of people in the auditorium at the start of the concert as found in part (a).

Sample: 2A
Score: 9

The student earned all 9 points.

Sample: 2B
Score: 6

The student earned 6 points: 2 points in part (a), 2 points in part (b), 2 points in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student earned the first 2 points by correctly computing $R'(t)$ and determining the correct interior critical point. The student considers the sign change of R' at $t = 1.363$, providing an argument for a local maximum instead of a global maximum, and did not earn the third point. In part (c) the student's work is correct. In part (d) the student computes the average value of $w'(t)$ over the interval from 1 to 2 instead of the total wait time $w(2)$ divided by the total number of people.

Sample: 2C
Score: 4

The student earned 4 points: 2 points in part (a), 1 point in part (b), 1 point in part (c), and no points in part (d). In part (a) the student's work is correct. In part (b) the student earned the first point for correctly computing $R'(t)$. The student's value for the critical point is incorrect, so the response was not eligible for the third point. In this case, no justification for a global maximum is given. In part (c) the student earned the first point for providing the correct definite integral for $w(2) - w(1)$. The student does not compute the value of the integral. In part (d) the student does not provide a definite integral for the numerator.

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2009 SCORING GUIDELINES

Question 3

Mighty Cable Company manufactures cable that sells for \$120 per meter. For a cable of fixed length, the cost of producing a portion of the cable varies with its distance from the beginning of the cable. Mighty reports that the cost to produce a portion of a cable that is x meters from the beginning of the cable is $6\sqrt{x}$ dollars per meter. (Note: Profit is defined to be the difference between the amount of money received by the company for selling the cable and the company's cost of producing the cable.)

(a) Find Mighty's profit on the sale of a 25-meter cable.

(b) Using correct units, explain the meaning of $\int_{25}^{30} 6\sqrt{x}\,dx$ in the context of this problem.

(c) Write an expression, involving an integral, that represents Mighty's profit on the sale of a cable that is k meters long.

(d) Find the maximum profit that Mighty could earn on the sale of one cable. Justify your answer.

(a) Profit $= 120 \cdot 25 - \int_0^{25} 6\sqrt{x}\,dx = 2500$ dollars	$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$
(b) $\int_{25}^{30} 6\sqrt{x}\,dx$ is the difference in cost, in dollars, of producing a cable of length 30 meters and a cable of length 25 meters.	1 : answer with units
(c) Profit $= 120k - \int_0^{k} 6\sqrt{x}\,dx$ dollars	$2: \begin{cases} 1: \text{integral} \\ 1: \text{expression} \end{cases}$
(d) Let $P(k)$ be the profit for a cable of length k. $P'(k) = 120 - 6\sqrt{k} = 0$ when $k = 400$. This is the only critical point for P, and P' changes from positive to negative at $k = 400$. Therefore, the maximum profit is $P(400) = 16{,}000$ dollars.	$4: \begin{cases} 1: P'(k) = 0 \\ 1: k = 400 \\ 1: \text{answer} \\ 1: \text{justification} \end{cases}$

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Work for problem 3(a)

For a 25-meter cable,

Total cost $= \int_0^{25} 6\sqrt{x}\,dx$

$= \$500.$

Total sell $= 120 \cdot 25 = \$3000$

Profit $= 3000 - 500$

$= \boxed{\$2500}$

Work for problem 3(b)

According to the problem, $\int_{25}^{30} 6\sqrt{x}\,dx$ represents the total cost in dollars of producing the part of a cable starting from 25 meters from the beginning of the cable to 30 meters from the beginning of the cable

Continue problem 3 on page 9.

Work for problem 3(c)

Profit =

$$P(k) = 120k - \int_0^k 6\sqrt{x}\,dx$$

$$= 120k - 6\int_0^k x^{\frac{1}{2}}\,dx$$

$$= 120k - 6\left(\frac{1}{\frac{3}{2}}x^{\frac{3}{2}} + C\right)$$

$$= 120k - \left(4x^{\frac{3}{2}} + C\right)$$

$\because$ when $x=0$, total cost $=0$, $\therefore C=0$

$\therefore \boxed{P(k) = 120k - 4k^{\frac{3}{2}}}$ is the equation

Work for problem 3(d)

According to the profit equation,

$P'(k) = 120 - 6k^{\frac{1}{2}}$, when $k=400$m, $P'(k)=0$ which is the only critical pt

Interval	$(0, 400)$	$(400, +\infty)$
Sign of p'	+	−
Behavior	inc.	dec.

Since p' changes sign from positive to negative at $k=400$ m, it is where the maximum occurs

when $k=400$m, $P(k) = \boxed{\$16000}$

END OF PART A OF SECTION II

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

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Work for problem 3(a)

revenue = \$120(25) = \$3,000

Cost = \$6 $\sqrt{25}$ (25) = \$750

profit = revenue - cost = \$3,000 - \$750 = $\boxed{\$2,250}$

Work for problem 3(b)

$\int_{25}^{30} 6\sqrt{x}\,dx$ is the total number of dollars it costs the company to produce the portion of a cable from 25 meters to 30 meters.

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Continue problem 3 on page 9.

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Work for problem 3(c)

$$P(k) = 120k - \int_0^k (6\sqrt{x})\,dx$$

Work for problem 3(d)

$$P'(k) = 120 - 6\sqrt{k} = 0$$

$$120 = 6\sqrt{k}$$

$$20 = \sqrt{k}$$

$$400 = k$$

$$P(400) = 120(400) - \int_0^{400} 6\sqrt{x}\,dx$$

$\boxed{\text{max profit} = \$16{,}000}$

$P'(k)$: + | − at 400

This is the max profit because the derivative of profit equals zero here and is changing from positive to negative.

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END OF PART A OF SECTION II
IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

3C1

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Work for problem 3(a)

production cost = $\$ 6\sqrt{x}$/m x = distance from end

price = $120/m

production(25) = $\int_0^{25} 6\sqrt{x}\,dx$

$\boxed{\approx 500 \text{ dollars}}$

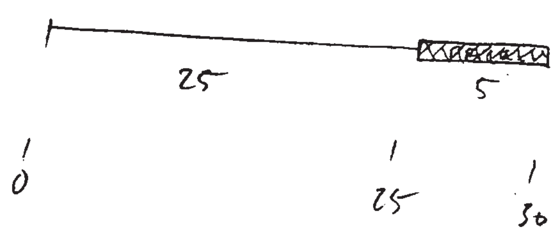
price(25) = 120/m = $120 · 25m

$\boxed{\approx 3000 \text{ dollars}}$

$\boxed{\text{profit} = \$2500}$

Work for problem 3(b)

$\int_{25}^{30} 6\sqrt{x}\,dx$ is the price of producing a 5m length of cable, starting 25m from the end (the shaded portion).



Continue problem 3 on page 9

Work for problem 3(c)

$$\boxed{\int_0^K 120x - 6\sqrt{x}\,dx}$$

4.

$.7 \times \cancel{70}$

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Work for problem 3(d)

$p(t) = \int_0^K 120x - 6\sqrt{x}\,dx$ $\quad x^{1/2}$

$p'(t) = 120K - 6\sqrt{K} = 0$

$6\sqrt{K} = 120K$

$\frac{6}{120} = \frac{K}{\sqrt{K}}$

$\frac{1}{2} = K^1(K^{.5})$

$\frac{1}{2} = K^{1/2}$

$\boxed{K \approx .7}$

$p'(t) = 120K - 6\sqrt{K}$

$p''(t) = 120 - 3/\sqrt{K}$

When $p'(t) = 0$ + $p''(t)$ is negative that is profit.

$p''(.7) = 120 - 3/\sqrt{.7}$

$120 - 3/$

END OF PART A OF SECTION II
IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY. DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

Question 3

Overview

This problem provided the context of a company, Mighty Cable, that manufactures and sells cables. Mighty sells cable for \$120 per meter, and the cost of producing the portion of a length of cable that is x meters from the beginning of the cable is reported to be $6\sqrt{x}$ dollars per meter. Part (a) asked for Mighty's profit on the sale of a 25-meter cable, which is defined to be the difference between the revenue from selling the cable and the cost to produce it. To calculate the cost to produce the cable, a student should have recognized that $6\sqrt{x}$ represents the rate of change of production cost for the cable with respect to the distance x from the beginning of the cable and that integrating this rate of change of cost gives the total cost to produce the cable. Part (b) asked students to interpret the definite integral $\int_{25}^{30} 6\sqrt{x}\,dx$ in the context of the problem. In part (c) students were asked to write an expression involving an integral that represents Mighty's profit on the sale of a k-meter cable, thus generalizing part (a) with the parameter k in place of the constant 25. Part (d) asked for the length k that maximizes profit, which required students either to apply the Fundamental Theorem of Calculus to a qualifying answer from part (d) or to recognize that the rate of change of profit with respect to length k is the difference of rates of change of income (\$120 per meter) and of production cost ($6\sqrt{k}$ dollars per meter).

Sample: 3A
Score: 9

The student earned all 9 points.

Sample: 3B
Score: 6

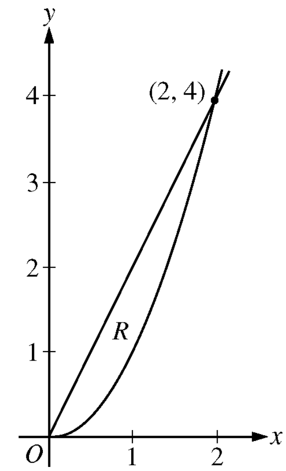
The student earned 6 points: no points in part (a), 1 point in part (b), 2 points in part (c), and 3 points in part (d). In part (a) the student makes no reference to an integral. In parts (b) and (c) the student's work is correct. In part (d) the student earned the first point with the equation $P'(k) = 120 - 6\sqrt{k} = 0.$ The student earned the second and third points by correctly solving for k and finding the maximum profit of \$16,000. The justification point was not earned because the student uses a local argument.

Sample: 3C
Score: 4

The student earned 4 points: 2 points in part (a), no points in part (b), 1 point in part (c), and 1 point in part (d). In part (a) the student's work is correct. The student earned the answer point but does not explicitly state $3000 - 500 = 2500.$ In part (b) the student does not use the unit of dollars. In part (c) the student earned the first point for presenting an expression involving $\int_0^k 6\sqrt{x}\,dx.$ The student incorrectly uses $120x$ in the integrand and did not earn the second point. In part (d) the student earned the first point with the equation $P'(k) = 120k - 6\sqrt{k} = 0.$ Although this profit equation is incorrect, the student was rewarded for correctly handling the imported incorrect expression from part (c). The student solves the equation incorrectly. The student's function does not have an absolute maximum, so the student was not eligible for additional points.

Question 4

Let R be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.



(a) Find the area of R.

(b) The region R is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \sin\left(\frac{\pi}{2}x\right)$. Find the volume of the solid.

(c) Another solid has the same base R. For this solid, the cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.

(a) $$\text{Area} = \int_0^2 \left(2x - x^2\right) dx = x^2 - \frac{1}{3}x^3 \Big|_{x=0}^{x=2} = \frac{4}{3}$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) $$\text{Volume} = \int_0^2 \sin\left(\frac{\pi}{2}x\right) dx = -\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\Big|_{x=0}^{x=2} = \frac{4}{\pi}$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) $$\text{Volume} = \int_0^4 \left(\sqrt{y} - \frac{y}{2}\right)^2 dy$$

$3 : \begin{cases} 2 : \text{integrand} \\ 1 : \text{limits} \end{cases}$

NO CALCULATOR ALLOWED

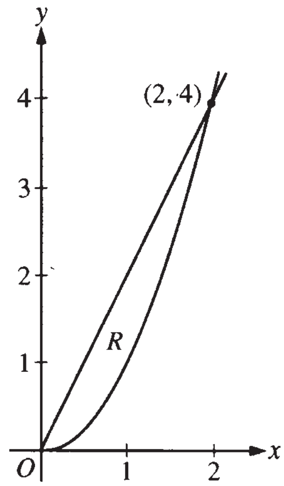
CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Work for problem 4(a)

$y = 2x \quad y = x^2$

$A = \int_0^2 (2x) - (x^2)\,dx$

$= x^2 - \frac{1}{3}x^3 \Big]_0^2$

$= 4 - \frac{1}{3}(8)$

$= 4 - \frac{8}{3} = \frac{12}{3} - \frac{8}{3} = \boxed{\frac{4}{3}}$

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Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

Work for problem 4(b)

$V = \int A(x)$

$V = \int_0^2 \left[\sin\left(\frac{\pi}{2}x\right)\right] dx \qquad u = \frac{\pi}{2}x$

$\frac{2}{\pi} du = \cancel{\frac{\pi}{2}} du$

$V = \frac{2}{\pi}\left[-\cos\frac{\pi}{2}x\right]_0^2$

$= \frac{2}{\pi}\left[-\cos\pi + \cos 0\right)$

$= \frac{2}{\pi}\left[1 + 1\right] = \boxed{\frac{4}{\pi}}$

Work for problem 4(c)

$V = \int A(x) \qquad A(x) = s^2$

$s = \sqrt{y} - \frac{1}{2}y$

$\boxed{V = \int_0^4 \left[\sqrt{y} - \frac{1}{2}y\right]^2 dy}$

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NO CALCULATOR ALLOWED

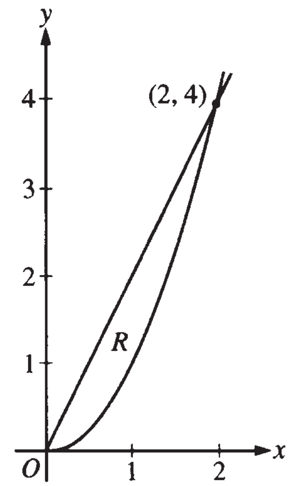
CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



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Work for problem 4(a)

$$\int_0^2 2x - x^2\,dx$$

$$x^2 - \frac{1}{3}x^3\Big]_0^2$$

$$\left(2^2 - \frac{1}{3}2^3\right) - \left(0^2 - \frac{1}{3}0^3\right)$$

$$4 - \frac{1}{3}\cdot 8$$

$$\frac{12}{3} - \frac{8}{3}$$

$$\boxed{\frac{4}{3}\text{ units}^2}$$

Do not write beyond this border.

Continue problem 4 on page 11.

NO CALCULATOR ALLOWED

Work for problem 4(b)

$$\int_0^2 \sin\left(\tfrac{\pi}{2}x\right) dx$$

$$-\cos\left(\tfrac{\pi}{2}x\right)\cdot \tfrac{\pi}{2} \Big]_0^2$$

$$-\tfrac{\pi}{2}\left[\cos \tfrac{\cancel{2}\pi}{\cancel{2}} - \cos 0\right]$$

$$-\tfrac{\pi}{2}\left[-1-1\right]$$

$$\cancel{-}\tfrac{\pi}{\cancel{2}}(\cancel{-}\cancel{2}) = \boxed{\pi \text{ units}^3}$$

Work for problem 4(c)

$$\int_0^2 s^2 dx$$

$$s = 2x - x^2$$

$$\int_0^2 (2x - x^2)^2 dx$$

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NO CALCULATOR ALLOWED

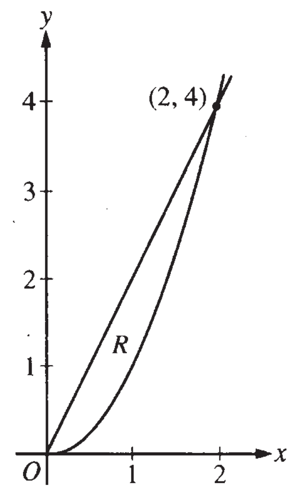
CALCULUS AB

SECTION II, Part B

Time—45 minutes

Number of problems—3

No calculator is allowed for these problems.



Work for problem 4(a)

Area of R = $\int_0^2 (2x)dx - \int_0^2 (x^2)dx$

$\left[(x^2) - \frac{1}{3}x^3\right]_0^2$

$4 - \frac{8}{3}$

$\frac{12}{3} - \frac{8}{3} = \frac{4}{3}$

$\boxed{\text{Area of R: } 1.667}$

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Continue problem 4 on page 11.

Work for problem 4(b)

$A(x) = \sin\left(\frac{\pi}{2}x\right)$

$\int_0^2 (A(x))dx = \text{volume}$

$\int_0^2 \sin\left(\frac{\pi}{2}x\right)dx$

$\int_0^2 \left[\frac{\pi}{2}\cos(\tfrac{\pi}{2}x) \ldots\right]$

$\frac{\pi}{2}\cos(\pi) = \boxed{\frac{\pi}{2}}$

Work for problem 4(c)

$A = \cancel{\ldots}\ s^2$

$V = \left(\sqrt{\sin\left(\frac{\pi}{2}x\right)}\right)^3$

$s = \sqrt{\sin\left(\frac{\pi}{2}x\right.}$

$\int_0^2 \left(\sin\frac{\pi}{2}x\right)^{3/2} dx$

$V = \left(\sqrt{\sin\left(\frac{\pi}{2}x\right)}\right)^3$

$\left(\left(\sin\frac{\pi}{2}x\right)^{\frac{1}{2}}\right)^3$

$\left(\sin\frac{\pi}{2}x\right)^{\frac{3}{2}}$

GO ON TO THE NEXT PAGE.

Question 4

Overview

Students were given the graph of a region R bounded by two curves in the xy-plane, $y = 2x$ and $y = x^2$. The points of intersection of the two curves were shown on the supplied graph. In part (a) students were asked to find the area of R, which required an appropriate integral (or difference of integrals), antiderivative, and evaluation. Part (b) asked students to find the volume of a solid whose cross-sectional area (perpendicular to the x-axis) at each x is given by $A(x) = \sin\left(\frac{\pi}{2}\right)$. Students had to set up the appropriate integral and find an antiderivative to evaluate the integral. Part (c) asked students to write, but not evaluate, an integral expression for the volume of a solid whose base is the region R and whose cross sections perpendicular to the y-axis are squares.

Sample: 4A
Score: 9

The student earned all 9 points.

Sample: 4B
Score: 6

The student earned 6 points: 3 points in part (a), 2 points in part (b), and 1 point in part (c). In part (a) the student's work is correct. In part (b) the student earned the first point for a correct integrand. The student's u-substitution is incorrect. The student was eligible for and earned the answer point. In part (c) the student's answer is correct for the volume of the solid with square cross sections perpendicular to the <u>x-axis</u>. This special case of $\int_0^2 \left(2x - x^2\right)^2\, dx$ earned 1 point.

Sample: 4C
Score: 3

The student earned 3 points: 2 points in part (a), 1 point in part (b), and no points in part (c). In part (a) the student earned the first 2 points. The student incorrectly reports the correct answer of $\frac{4}{3}$ as 1.667. In part (b) the student earned the first point for a correct integrand. The student's u-substitution is incorrect, and the student was not eligible for the answer point since $\frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right)\Big|_0^2$ is negative. In part (c) the student did not earn any points for the integrand since it is not of the form $(f(y) - g(y))^2$. The student's limits are incorrect.

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Question 5

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.

(a) Estimate $f'(4)$. Show the work that leads to your answer.

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\, dx$. Show the work that leads to your answer.

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\, dx$. Show the work that leads to your answer.

(d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

Solution	Points
(a) $f'(4) \approx \dfrac{f(5) - f(3)}{5 - 3} = -3$	1 : answer
(b) $\int_2^{13} (3 - 5f'(x))\, dx = \int_2^{13} 3\, dx - 5\int_2^{13} f'(x)\, dx = 3(13 - 2) - 5(f(13) - f(2)) = 8$	2 : { 1 : uses Fundamental Theorem of Calculus; 1 : answer }
(c) $\int_2^{13} f(x)\, dx \approx f(2)(3 - 2) + f(3)(5 - 3) + f(5)(8 - 5) + f(8)(13 - 8) = 18$	2 : { 1 : left Riemann sum; 1 : answer }
(d) An equation for the tangent line is $y = -2 + 3(x - 5)$. Since $f''(x) < 0$ for all x in the interval $5 \le x \le 8$, the line tangent to the graph of $y = f(x)$ at $x = 5$ lies above the graph for all x in the interval $5 < x \le 8$. Therefore, $f(7) \le -2 + 3 \cdot 2 = 4$. An equation for the secant line is $y = -2 + \frac{5}{3}(x - 5)$. Since $f''(x) < 0$ for all x in the interval $5 \le x \le 8$, the secant line connecting $(5, f(5))$ and $(8, f(8))$ lies below the graph of $y = f(x)$ for all x in the interval $5 < x < 8$. Therefore, $f(7) \ge -2 + \frac{5}{3} \cdot 2 = \frac{4}{3}$.	4 : { 1 : tangent line; 1 : shows $f(7) \le 4$; 1 : secant line; 1 : shows $f(7) \ge \frac{4}{3}$ }

5 5 5 5 5 5 5 5 5 5

NO CALCULATOR ALLOWED

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

5A1

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Work for problem 5(a)

$$f'(4) \approx \frac{-2-4}{5-3} = \frac{-6}{2} = -3$$

Work for problem 5(b)

$$\int_2^{13} 3 - 5f'(x)\,dx = \int_2^{13} 3\,dx - 5\int_2^{13} f'(x)\,dx$$

$$3x\Big|_2^{13} - 5\left(f(13) - f(2)\right)$$

$$33 - 5(6-1)$$

$$33 - 5(5)$$

$$33 - 25 = 8$$

NO CALCULATOR ALLOWED

5A2

Work for problem 5(c)

~~1·1+2·4+~~

$(3-2)\cdot 1+(5-3)\cdot 4+(8-5)\cdot -2+(13-8)\cdot 3$

$1+8-6+15=18$

Work for problem 5(d)

$y+2=3(x-5)$

$y+2=3(7-5)$

$y+2=3(2)$

$y=4$

since $f''(x)<0$ the tangent line is an overapproximation so $f(7)\leq 4$

$\frac{3--2}{8-5}=\frac{5}{3}$

$y+2=\frac{5}{3}(x-5)$

$y+2=\frac{5}{3}(7-5)$

$y+2=\frac{5}{3}(2)$

$y+2=\frac{10}{3}$

$y=\frac{10}{3}-2$

$y=\frac{4}{3}$

since $f''(x)<0$ the secant line is an underapproximation at $f(7)$, so $f(7)$ must be $\geq\frac{4}{3}$

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NO CALCULATOR ALLOWED

5B1

x	2	3	5	8	13
$f(x)$	1	4	–2	3	6

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Work for problem 5(a)

$$f'(4) = \frac{f(5)-f(3)}{5-3}$$

$$f'(4) = \frac{-2-4}{5-3}$$

$$f'(4) = \frac{-6}{2} = -3$$

Work for problem 5(b)

$$\int_2^{13} (3 - 5f'(x))\,dx$$

$$\int_2^{13} 3\,dx - 5\int_2^{13} f'(x)\,dx$$

$$3x\Big|_2^{13} - 5\left(f(x)\Big|_2^{13}\right)$$

$$3(13) - 3(2) - 5(f(13) - f(2))$$

$$26 - 6 - 5(6-1)$$

$$20 - 5(5)$$

$$20 - 25$$

$$\int_2^{13} (3 - 5f'(x))\,dx = 5$$

NO CALCULATOR ALLOWED

5B2

Work for problem 5(c)

$\int_2^{13} f(x)\,dx$ using left Riemann sums:

$(1\cdot 2)+(2\cdot 4)+(3\cdot -2)+(5\cdot 3)$

$2+8-6+15=19$

$\int_2^{13} f(x)\,dx \approx 19$

Work for problem 5(d)

$f'(5)=3$ and $f(5)=-2$

$y+2=3(x-5)$

$y=3x-15-2$

$y=3x-17$

at $x=7$: $y=3(7)-17$

$y=21-17$

$y=4$

Because $f''(x)<0$, $f'(x)$ is decreasing over the interval $5\le x\le 8$. This means $f'(5)$ is the largest value over this interval, so $f(7)$ can not be any greater than 4. Therefore, $f(7)\le 4$.

$f'(c)=\dfrac{f(8)-f(5)}{\cancel{8-5}}$

$f'(c)=\dfrac{3--2}{3}=\dfrac{5}{3}$

$y+2=\frac{5}{3}(x-5)$

at $x=7$: $y+2=\frac{5}{3}(7-5)$

$y+2=\frac{10}{3}$

$y=\frac{4}{3}$

Because this secant line is the average slope of the interval $5\le x\le 8$, it is an underapproximation of $f(7)$. Therefore $f(7)\ge \frac{4}{3}$.

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NO CALCULATOR ALLOWED

5C1

x	2	3	5	8	13
$f(x)$	1	4	−2	3	6

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Work for problem 5(a)

$f'(4) = \frac{-2-1}{5-2} = -\frac{3}{3} = -1$

$\boxed{f'(4) = -1}$

Work for problem 5(b)

$\int_2^{13} (3 - 5f'(x))\,dx$

$\Big|_2^{13}\ 3x - 5(f(x))$

$13(3) - 6$

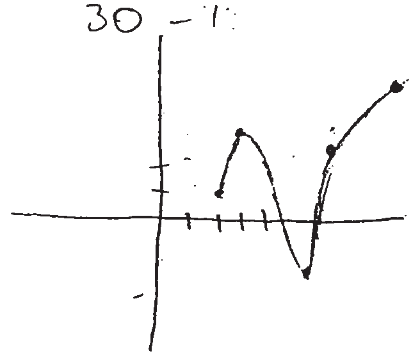
$39 - 6 = 33$

$33 - 29 = 4$

$\Big|_2^{13}\ 5f(x)$

$5(f(13)) - 5(f(2))$

$30 - 1$



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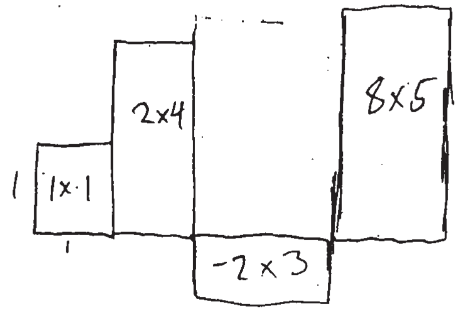
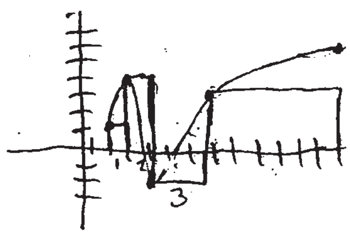
NO CALCULATOR ALLOWED

5C8

Work for problem 5(c)



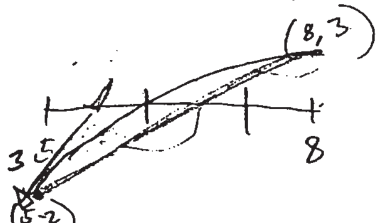


$1+8-6+40$

$9-6$ 43

Work for problem 5(d)

$f'(5)=3$



$\frac{3+2}{8-5}=\frac{5}{3}$

$y+2=3(x-5)$

$y+2=3x-15$

$y=3x-17$

$y=3(7)-17$

$y=21-17$

$y=4$ ≤ 4

$f(7)$

$y+2=\frac{5}{3}(x-5)$

$y+2=\frac{5x}{3}-\frac{25}{3}$

$y=\frac{5x}{3}-\frac{31}{3}$

$f(7)\geq \frac{4}{3}$

$y=\frac{35}{3}-\frac{31}{3}=$ $\frac{4}{3}\geq\frac{4}{3}$

Do not write beyond this border.

Question 5

Overview

This problem presented students with a table of values for a function f sampled at five values of x. It was also stated that f is twice differentiable for all real numbers. Part (a) asked for an estimate for $f'(4)$. Since $x = 4$ falls between the values sampled on the table, students should have calculated the slope of the secant line to the graph of f corresponding to the closest pair of points in the supplied data that brackets $x = 4$. Part (b) tested students' ability to apply properties of the definite integral to evaluate $\int_2^{13} (3 - 5f'(x))\,dx$. Part (c) asked for an approximation to $\int_2^{13} f(x)\,dx$ using the subintervals of $[2, 13]$ indicated by the data in the table. In part (d) it was also stated that $f'(5) = 3$ and $f''(x) < 0$ for all x in $[5, 8]$. Students were asked to use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$ and to use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$. For the former inequality, students should have used the fact that f'' is negative (so f' is decreasing) on $[5, 8]$ so that the tangent line at $x = 5$ lies above the graph of f throughout $(5, 8]$. For the latter inequality, students should have used the sign of f'' to conclude that the indicated secant line lies below the graph of f for $5 < x < 8$; in particular, the point on the graph of the secant line corresponding to $x = 7$ is below the corresponding point on the graph of f.

Sample: 5A
Score: 9

The student earned all 9 points. In part (b) the student's second line earned the first point, and the third line earned the second point. In part (c) the student's first line earned both points. In part (d) the student's first line earned the first point. The second point was earned by showing that $y = 4$ when $x = 7$ on the tangent line, stating the desired inequality $f(7) \le 4$, and giving an acceptable reason to validate the inequality. The third and fourth points were earned in a similar manner using the secant line.

Sample: 5B
Score: 6

The student earned 6 points: 1 point in part (a), 1 point in part (b), 1 point in part (c), and 3 points in part (d). In part (a) the student's second line earned the point. In part (b) the student's fourth line earned the first point for use of the Fundamental Theorem of Calculus. The student makes subsequent errors. In part (c) the student's second line earned the first point since seven of the eight presented factors are correct. The student did not earn the answer point. In part (d) the student's second line on the left earned the first point. The second point was earned by showing that $y = 4$ when $x = 7$ on the tangent line, stating the desired inequality $f(7) \le 4$, and giving an acceptable reason to validate the inequality. The student's third line on the right earned the third point. The last point was not earned since the student's reason does not validate the inequality $f(7) \ge \frac{4}{3}$.

Question 5 (continued)

Sample: 5C
Score: 4

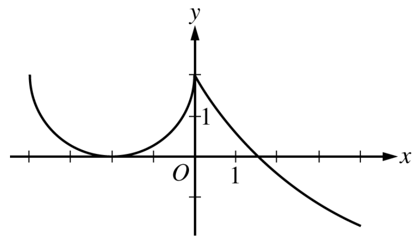
The student earned 4 points: no points in part (a), 1 point in part (b), 1 point in part (c), and 2 points in part (d). In part (a) the student's answer is incorrect. In part (b) the student earned the first point by correctly applying the Fundamental Theorem of Calculus to the derivative of f. The student makes a subsequent arithmetic error. In part (c) the student earned the first point since seven of the eight presented factors are correct. The student did not earn the answer point. In part (d) the student earned the first and third points for correct equations for the tangent and secant lines. Since the student does not explain why either of the two inequalities is valid, the student did not earn the other points.

Question 6

The derivative of a function f is defined by

$$f'(x) = \begin{cases} g(x) & \text{for } -4 \le x \le 0 \\ 5e^{-x/3} - 3 & \text{for } 0 < x \le 4 \end{cases}.$$

The graph of the continuous function f', shown in the figure above, has x-intercepts at $x = -2$ and $x = 3\ln\left(\frac{5}{3}\right)$. The graph of g on $-4 \le x \le 0$ is a semicircle, and $f(0) = 5$.



Graph of f'

(a) For $-4 < x < 4$, find all values of x at which the graph of f has a point of inflection. Justify your answer.

(b) Find $f(-4)$ and $f(4)$.

(c) For $-4 \le x \le 4$, find the value of x at which f has an absolute maximum. Justify your answer.

(a) f' changes from decreasing to increasing at $x = -2$ and from increasing to decreasing at $x = 0$. Therefore, the graph of f has points of inflection at $x = -2$ and $x = 0$.

$2 : \begin{cases} 1 : \text{identifies } x = -2 \text{ or } x = 0 \\ 1 : \text{answer with justification} \end{cases}$

(b) $f(-4) = 5 + \int_0^{-4} g(x)\, dx$

$\qquad = 5 - (8 - 2\pi) = 2\pi - 3$

$f(4) = 5 + \int_0^4 \left(5e^{-x/3} - 3\right) dx$

$\qquad = 5 + \left(-15e^{-x/3} - 3x\right)\Big|_{x=0}^{x=4}$

$\qquad = 8 - 15e^{-4/3}$

$5 : \begin{cases} 2 : f(-4) \\ \quad 1 : \text{integral} \\ \quad 1 : \text{value} \\ 3 : f(4) \\ \quad 1 : \text{integral} \\ \quad 1 : \text{antiderivative} \\ \quad 1 : \text{value} \end{cases}$

(c) Since $f'(x) > 0$ on the intervals $-4 < x < -2$ and $-2 < x < 3\ln\left(\frac{5}{3}\right)$, f is increasing on the interval $-4 \le x \le 3\ln\left(\frac{5}{3}\right)$.

Since $f'(x) < 0$ on the interval $3\ln\left(\frac{5}{3}\right) < x < 4$, f is decreasing on the interval $3\ln\left(\frac{5}{3}\right) \le x \le 4$.

Therefore, f has an absolute maximum at $x = 3\ln\left(\frac{5}{3}\right)$.

$2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{justification} \end{cases}$

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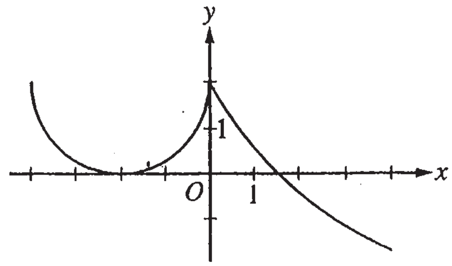

Graph of f'

Work for problem 6(a)

The points of inflection are at

$x=-2$ and $x=0$

at $x=-2$ f' decreases then increases

at $x=0$ f' increases then decreases

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Work for problem 6(b)

$$f(-4) = 5 - \int_{-4}^{0} g(x)\,dx$$
$$= 5 - \left[4(2) - \tfrac{1}{2}\pi(2)^2\right]$$
$$= 5 - \left[8 - 2\pi\right] = 2\pi - 3$$

$$f(4) = 5 + \int_{0}^{4} (5e^{-x/3} - 3)\,dx$$
$$= 5 + \left[-3 \cdot 5e^{-x/3} - 3x\right]_0^4$$
$$= 5 + \left[-15e^{-4/3} - 12 + 15\right]$$
$$= 5 - 15e^{-4/3} + 3$$
$$= 8 - 15e^{-4/3}$$

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Work for problem 6(c)

The absolute maximum is at $x = 3\ln 5/3$ since $f' > 0$ on the interval $[-4, -2)$ and $(-2, 3\ln 5/3)$ which means f is increasing on those intervals also $f' < 0$ on the interval $(3\ln 5/3, 4]$ which means f is decreasing on this interval.

6

6B1

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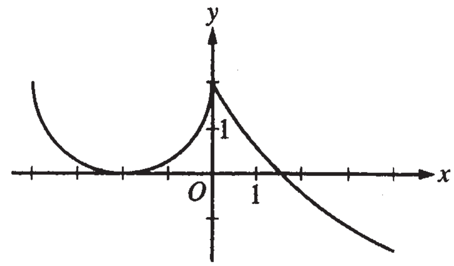

Graph of f'

Work for problem 6(a)

$f'(x)$ ← dec. inc. decrea → −4 −2 0 $3\ln(\frac{5}{3})$ 4

$f''(x)$ ← − + − → −4 −2 0 4

f(x) has a point of inflection at x = −2 and x = 0 because f''(x) changes from ⊖ to ⊕ at x = −2 and f''(x) changes from ⊕ to ⊖ at x = 0

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Continue problem 6 on page 15.

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Work for problem 6(b)

$f(-4) = \int_0^{-4} f'(x)\,dx$

$\frac{\pi r^2}{2} = \frac{\pi 4}{2} = 2\pi$

$2(4) = 8$

$\boxed{8 - 2\pi}$

$f(4) = \int_0^4 f'(x)\,dx = \int_0^4 5e^{-x/3} - 3\,dx$

$= -15e^{\frac{-x}{3}} - 3x \Big|_0^4$

$= -15e^{\frac{-4}{3}} - 12 - (-15 - 0)$

$= \boxed{-15e^{\frac{-4}{3}} + 3}$

Work for problem 6(c)

f(x) has an absolute maximum at $x = 3\ln\left(\frac{5}{3}\right)$ because $f'(x) = 0$ at $x = 3\ln\left(\frac{5}{3}\right)$ and changes from + to − at $x = 3\ln\left(\frac{5}{3}\right)$

GO ON TO THE NEXT PAGE.

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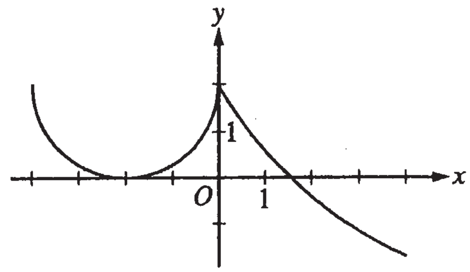
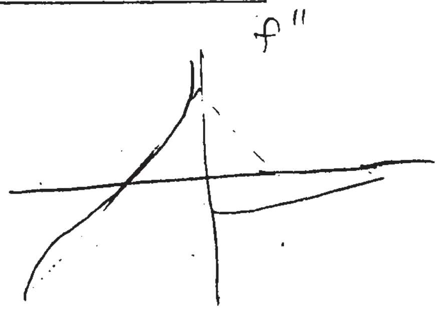


Graph of f'

Work for problem 6(a)



at -2 there is an inflection point.
-2 is where f' =0 and f'' changes sign.

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Work for problem 6(b)

$f(-4) = \int_{-4}^{0} g(x)$

$f(4) = \int_0^4 f'(x)\,dx = \int_0^4 5e^{-x/3} - 3\,dx$

$-15e^{-x/3}\Big|_0^4 - 3x\Big|_0^4$

$-15e^{-4/3} + 15e^0 - 12$

$f(4) = -15e^{-4/3} + 3$

$f(-4) = 8 - \frac{1}{2}\pi r^2 \qquad r = 2$

$f(-4) = 8 - 2\pi$

Work for problem 6(c)

at $x = 1.5$, there is an absolute maximum

from $(-4, 1.5)$, $f(x)$ is increasing

at $f(1.5)$, $f'(x)$ sign changes, making $f(x)$ decrease

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Question 6

Overview

In this problem a function f satisfies $f(0) = 5$ and has continuous first derivative for $-4 \le x \le 4$. The graph of f' was supplied. For $-4 \le x \le 0$, the graph of f' is a semicircle tangent to the x-axis at $x = -2$ and tangent to the y-axis at $y = 2$. For $0 < x \le 4$, $f'(x) = 5e^{-x/3} - 3$. Part (a) asked for those values of x in the interval $-4 < x < 4$ at which the graph of f has a point of inflection; these correspond to points where the graph of f' changes from increasing to decreasing, or vice versa. In part (b) students had to use the given initial value for f and the appropriate piece of f' to find $f(-4)$ and $f(4)$. The former value required the evaluation of an integral using geometry, and the latter required the evaluation of an integral via an antiderivative. Part (c) asked for the value of x at which f attains its absolute maximum on the interval $[-4, 4]$. Using the derivative of f, students should have concluded that f is increasing on $\left[-4, 3\ln\left(\frac{5}{3}\right)\right]$ and decreasing on $\left[3\ln\left(\frac{5}{3}\right), 4\right]$, so that the maximum must occur at $x = 3\ln\left(\frac{5}{3}\right)$.

Sample: 6A
Score: 9

The student earned all 9 points.

Sample: 6B
Score: 6

The student earned 6 points: 2 points in part (a), 3 points in part (b), and 1 point in part (c). In part (a) the student's work is correct. In part (b) the student incorrectly uses $f'(x)$ as the integrand in the definite integral for $f(-4)$. However, the student earned the integral point by giving the correct geometric evaluation of the integral as $8 - 2\pi$. The student also earned the first 2 points for $f(4)$. The student did not earn either value point. In part (c) the student gives the correct absolute maximum. The justification point was not earned since the student does not provide a global argument.

Sample: 6C
Score: 4

The student earned 4 points: 1 point in part (a), 3 points in part (b), and no points in part (c). In part (a) the student earned the first point for one correct point of inflection but was not eligible for the justification point. In part (b) the student earned the integral point for $f(-4)$ as well as the first 2 points for $f(4)$. The student did not earn either value point. In part (c) the student's work is incorrect. The student estimates the x-intercept as 1.5 instead of using the information given in the question.